

# Tangent space to Milnor $K$ -groups of rings

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## Abstract

We prove that the tangent space to the  $(n+1)$ -th Milnor  $K$ -group of a ring  $R$  is isomorphic to group of  $n$ -th absolute Kähler differentials of  $R$  when the ring  $R$  contains  $\frac{1}{2}$  and has sufficiently many invertible elements. More precisely, the latter condition is that  $R$  is weakly 5-fold stable in the sense of Morrow.

## 1 Introduction

The Milnor  $K$ -group  $K_n^M(R)$  of a commutative associative unital ring  $R$  is generated by symbols  $\{r_1, \dots, r_n\}$ ,  $r_i \in R^*$ , that satisfy the Steinberg relations (see Definition 2.1). Studying Milnor  $K$ -groups, one often requires that  $R$  has sufficiently many invertible elements. In this context, van der Kallen [9] has introduced the notion of a  $k$ -fold stable ring for a natural number  $k$  (see Remark 2.4(i)). Recently Morrow [11] defined weakly  $k$ -fold stable rings (see Definition 2.2). Note that for any commutative associative unital ring  $A$ , the ring of Laurent series  $A((t))$  is weakly  $k$ -fold stable for all  $k$ , while for many natural rings  $A$ , the ring  $A((t))$  is not  $k$ -fold stable for any  $k$  (see Remark 2.4(ii)).

Let  $\varepsilon$  be a formal variable such that  $\varepsilon^2 = 0$ . By a tangent space  $TK_n^M(R)$  to Milnor  $K$ -group, we mean the kernel of the natural homomorphism  $K_n^M(R[\varepsilon]) \rightarrow K_n^M(R)$  (see Definition 2.5). Let  $\Omega_R^n$  denote the group of  $n$ -th absolute Kähler differentials of  $R$ . Following Bloch [1], one constructs a natural homomorphism (see Definition 2.7)

$$B : TK_{n+1}^M(R) \longrightarrow \Omega_R^n.$$

In particular, for any collection of invertible elements  $r_1, \dots, r_n \in R^*$  and any element  $s \in R$ , the homomorphism  $B$  sends the symbol  $\{1 + sr_1 \dots r_n \varepsilon, r_1, \dots, r_n\}$  to the differential form  $sdr_1 \wedge \dots \wedge dr_n$  (see Example 2.8).

The aim of the paper is to prove that  $B$  is an isomorphism when  $R$  contains  $\frac{1}{2}$  and is weakly 5-fold stable (see Theorem 2.9).

Besides being of independent interest, this statement is also of utmost importance for the proof of the explicit formula and the universal property of the higher-dimensional Contou-Carrère symbol. The proof will be given in the work [7] (the explicit formula was also announced in the short note [6]). In this proof, Theorem 2.9 is applied to the ring of iterated Laurent series  $R = A((t_1)) \dots ((t_n))$  over a ring  $A$ . Thus it is important that

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Theorem 2.9 is valid for any weakly 5-fold stable ring and not only for a 5-fold stable ring.

Let us explain how to deduce Theorem 2.9 in a weaker form from previously known results of van der Kallen and Bloch. Namely, in [8] it was constructed a natural isomorphism  $TK_2(R) \simeq \Omega_R^1$  for any ring  $R$  that contains  $\frac{1}{2}$ , where  $TK_2(R)$  is the tangent space to the algebraic  $K$ -group  $K_2(R)$ . Later it was proved in [9, Theor. 7.1, 8.4] that for a 5-fold stable ring  $R$ , there is an isomorphism  $K_2^M(R) \simeq K_2(R)$ , whence  $TK_2^M(R) \simeq TK_2(R)$ . The resulting isomorphism  $TK_2^M(R) \simeq \Omega_R^1$  coincides with  $B$ . This proves Theorem 2.9 when  $n = 1$  and  $R$  is a 5-fold stable ring that contains  $\frac{1}{2}$ . Using the case  $n = 1$  and closely following the strategy from [1], it is possible to obtain Theorem 2.9 when  $n$  is an arbitrary natural number and  $R$  is a 5-fold stable ring that contains  $\frac{1}{2}$  (but not a weakly 5-fold stable ring as in Theorem 2.9 itself). See also a recent paper of Dribus [4] for a more general statement about 5-fold stable rings, which is an analog for Milnor  $K$ -groups of the famous theorem of Goodwillie [5].

Notice that the above approach to a weaker form of Theorem 2.9 is based on the machinery of algebraic  $K$ -theory, in particular, on hard results of van der Kallen [9] about elements in the Steinberg extension. However, the theorem is essentially a statement about relations between symbols in Milnor  $K$ -groups  $K_{n+1}^M(R[\varepsilon])$  and it is natural to expect that there exists a direct proof in terms of symbols only. In this paper, we give such proof.

We prove Theorem 2.9 following the strategy from [1] but replacing van der Kallen's results on the algebraic  $K$ -group  $K_2$  by explicit calculations with symbols in the Milnor  $K$ -group  $K_2^M(R[\varepsilon])$  (see Lemmas 3.3 and 3.4). We believe that these calculations have their own interest as well.

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## 2 Statement of the main result

For short, by a ring we mean a commutative associative unital ring and by a *group functor*, we mean a covariant functor from the category of commutative associative unital rings to the category of abelian groups. Fix a natural number  $n \geq 0$ .

### 2.1 Milnor $K$ -groups

Recall the following well-known definition.

**Definition 2.1.** The  $n$ -th Milnor  $K$ -group  $K_n^M(R)$  of a ring  $R$  is the  $n$ -th homogenous component of the graded ring

$$\bigoplus_{n \geq 0} K_n^M(R) := \bigoplus_{n \geq 0} (R^*)^{\otimes n} / \text{St},$$

where  $\text{St}$  is the homogenous ideal generated by all elements of type  $r \otimes (1-r) \in R^* \otimes_{\mathbb{Z}} R^*$ .

We denote the group law in  $K_n^M(R)$  additively. Explicitly,  $K_n^M(R)$  is the quotient group of the group  $(R^*)^{\otimes n}$  by the subgroup generated by all elements of type

$$r_1 \otimes \dots \otimes r_i \otimes r \otimes (1 - r) \otimes r_{i+3} \otimes \dots \otimes r_n,$$

which are called *Steinberg relations*. Note that in this tensor,  $r$  and  $1 - r$  come one after another and are not separated. The class in  $K_n^M(R)$  of a tensor  $r_1 \otimes \dots \otimes r_n$ , where  $r_i \in R^*$ , is denoted by  $\{r_1, \dots, r_n\}$  and is usually called a *symbol*. Thus we do not require additional relations on symbols besides the multilinearity and the Steinberg relations. In particular, we have that  $K_0^M(R) = \mathbb{Z}$  and  $K_1^M(R) = R^*$ .

Clearly,  $K_n^M$  is a group functor. Denote by  $\Omega^n$  the group functor that sends a ring  $R$  to the group of  $n$ -th absolute Kähler differentials  $\Omega_R^n$ . It is easy to check that there is a morphism of group functors

$$d \log : K_n^M \longrightarrow \Omega^n, \quad \{r_1, \dots, r_n\} \longmapsto \frac{dr_1}{r_1} \wedge \dots \wedge \frac{dr_n}{r_n}.$$

## 2.2 Weakly stable rings

Recall the following definition given by Morrow in [11, Def. 3.1] (this is a slightly different form, which is equivalent to the one from op. cit.).

**Definition 2.2.** Given a natural number  $k \geq 2$ , a ring  $R$  is called *weakly  $k$ -fold stable* if for any collection of elements  $r_1, \dots, r_{k-1} \in R$ , there is an invertible element  $r \in R^*$  such that the elements  $r_1 + r, \dots, r_{k-1} + r$  are invertible in  $R$ .

*Example 2.3.*

- (i) A ring  $R$  is weakly 2-fold stable if and only if any element from  $R$  is a sum of two invertible elements.
- (ii) A semi-local ring is weakly  $k$ -fold stable if and only if each of its residue fields contains at least  $k + 1$  elements, [11, Rem. 3.3].
- (iii) For any ring  $A$ , the ring of Laurent series  $A((t)) = A[[t]][t^{-1}]$  is weakly  $k$ -fold stable for all  $k \geq 2$ . In fact, one can take an invertible element  $r$  in Definition 2.2 to be equal to an element  $t^i$  for a suitable  $i \in \mathbb{Z}$ .

*Remark 2.4.*

- (i) Let  $k \geq 1$ . Recall from [9] that a ring  $R$  is  *$k$ -fold stable* if for any collection of elements  $r_1, s_1, \dots, r_k, s_k$  with  $r_i, s_i \in R$  such that

$$r_1 R + s_1 R = \dots = r_k R + s_k R = R,$$

there is an element  $r \in R$  such that  $r_1 + rs_1, \dots, r_k + rs_k \in R^*$ . Note that if  $k \geq 2$ , then  $k$ -fold stability implies  $(k - 1)$ -fold stability and also implies weak  $k$ -fold stability, [11, §3.1].

- (ii) Following the same idea as in [11, Rem.3.5], we observe that the ring of Laurent series  $A((t))$  can be even not 1-fold stable and hence not  $k$ -fold stable for any  $k \geq 1$ . Namely, suppose that  $\text{Spec}(A)$  is connected,  $A$  has no nilpotent elements and has a non-invertible element  $a \in A$ . Then the pair  $a, at^{-1} + 1$  breaks the 1-fold stability for  $A((t))$ , that is, there is no a Laurent series  $f \in A((t))$  such that  $a + f(at^{-1} + 1) \in A((t))^*$ . Indeed, one shows that the first non-zero coefficient in  $a + f(at^{-1} + 1)$  belongs to the ideal  $(a) \subset A$ . Hence this coefficient is not invertible in  $A$ . Recall that for  $A$  as above, a Laurent series in  $A((t))$  is invertible if and only if its first non-zero coefficient is invertible, see [2, Lem. 1.3] and [3, Lem. 0.8]. This implies that  $a + f(at^{-1} + 1)$  is not invertible.

## 2.3 Tangent spaces to group functors

Below  $\varepsilon$  denotes a formal variable that satisfies  $\varepsilon^2 = 0$ . Thus for any ring  $R$ , we have an isomorphism  $R[\varepsilon] \simeq R[x]/(x^2)$ , where  $x$  is a formal variable.

**Definition 2.5.** Given a group functor  $F$ , a *tangent space*  $TF$  to  $F$  is the group functor

$$TF(R) := \text{Ker}(F(R[\varepsilon]) \rightarrow F(R)).$$

In particular, there is a decomposition  $F(R[\varepsilon]) \simeq F(R) \times TF(R)$ .

*Example 2.6.*

- (i) We have that  $d(\varepsilon^2) = 2\varepsilon d\varepsilon = 0$  in  $\Omega_{R[\varepsilon]}^1$  and a calculation shows that there is an isomorphism of  $R$ -modules

$$T\Omega^{n+1}(R) \simeq (\varepsilon \Omega_R^{n+1}) \oplus (d\varepsilon \wedge \Omega_R^n) \oplus ((\varepsilon d\varepsilon \wedge \Omega_R^n)/2(\varepsilon d\varepsilon \wedge \Omega_R^n)). \quad (1)$$

In particular, if  $\frac{1}{2} \in R$ , then the last summand equals zero.

- (ii) There is a group decomposition  $R[\varepsilon]^* \simeq R^* \times (1 + R\varepsilon)$ . It follows that the subgroup  $TK_{n+1}^M(R) \subset K_{n+1}^M(R)$  is generated by symbols  $\{u_1, \dots, u_n\}$ , where each element  $u_i \in R[\varepsilon]^*$  is either from the subgroup  $R^*$  or from the subgroup  $1 + R\varepsilon$ , and at least one  $u_i$  is from  $1 + R\varepsilon$ .

Following a construction of Bloch [1], we give the next definition.

**Definition 2.7.** Denote by

$$B : TK_{n+1}^M \longrightarrow \Omega^n$$

the morphism of group functors obtained as the composition of  $d \log : TK_{n+1}^M \rightarrow T\Omega^{n+1}$  and the projection to the direct summand  $\Omega^n \simeq d\varepsilon \wedge \Omega^n$  in decomposition (1).

*Example 2.8.* For any collection of invertible elements  $r_1, \dots, r_n \in R^*$  and any element  $s \in R$ , there is an equality

$$B\{1 + sr_1 \dots r_n \varepsilon, r_1, \dots, r_n\} = sdr_1 \wedge \dots \wedge dr_n.$$

Indeed, we have that

$$\begin{aligned} \frac{d(1 + sr_1 \dots r_n \varepsilon)}{1 + sr_1 \dots r_n \varepsilon} &= (1 - sr_1 \dots r_n \varepsilon) d(sr_1 \dots r_n \varepsilon) = \\ &= \varepsilon d(sr_1 \dots r_n) + sr_1 \dots r_n d\varepsilon - (sr_1 \dots r_n)^2 \varepsilon d\varepsilon. \end{aligned}$$

Therefore, there are equalities

$$\begin{aligned} d \log \{1 + sr_1 \dots r_n \varepsilon, r_1, \dots, r_n\} &= \frac{d(1 + sr_1 \dots r_n \varepsilon)}{1 + sr_1 \dots r_n \varepsilon} \wedge \frac{dr_1}{r_1} \wedge \dots \wedge \frac{dr_n}{r_n} = \\ &= \varepsilon \left( d(sr_1 \dots r_n) \wedge \frac{dr_1}{r_1} \wedge \dots \wedge \frac{dr_n}{r_n} \right) + d\varepsilon \wedge (sdr_1 \wedge \dots \wedge dr_n) - \\ &\quad - \varepsilon d\varepsilon \wedge (s^2 r_1 \dots r_n dr_1 \wedge \dots \wedge dr_n). \end{aligned}$$

The projection of this element to the direct summand  $\Omega_R^n \simeq d\varepsilon \wedge \Omega_R^n$  in decomposition (1) equals  $sdr_1 \wedge \dots \wedge dr_n$ .

Here is the main result of the paper.

**Theorem 2.9.** *Let  $R$  be a ring such that  $R$  contains  $\frac{1}{2}$  and is weakly 5-fold stable. Then the homomorphism  $B: TK_{n+1}^M(R) \rightarrow \Omega_R^n$  is an isomorphism.*

We do not know whether Theorem 2.9 is true for weakly 4-fold stable rings containing  $\frac{1}{2}$ .

### 3 Proof of the main result

As above,  $n \geq 0$  is a natural number and  $\varepsilon$  denotes a formal variable that satisfies the relation  $\varepsilon^2 = 0$ .

#### 3.1 Auxiliary results

We start with the following elementary lemma.

**Lemma 3.1.** *Let  $G$  be an Abelian group. Suppose there exists an automorphism  $\varphi: G \rightarrow G$  such that  $G$  is generated by elements  $g \in G$  that satisfy  $\varphi(g) = 2^i \cdot g$  for some natural number  $i \geq 1$  depending on  $g$ . Then the group  $G$  is uniquely 2-divisible.*

*Proof.* We need to show that multiplication by 2 is a bijection from  $G$  to itself. Define the following increasing filtration on  $G$ : put  $F_0 G = \{0\}$  and let  $F_l G$ ,  $l \geq 1$ , be the subgroup generated by elements  $g \in G$  that satisfy  $\varphi(g) = 2^i \cdot g$  for some natural number  $i$ ,  $1 \leq i \leq l$ . It is enough to show that multiplication by 2 is a bijection on each adjoint quotient  $F_l G / F_{l-1} G$ ,  $l \geq 1$ . It is easily seen that the filtration is preserved by any endomorphism of the group  $G$  that commutes with  $\varphi$ . In particular, the filtration is preserved by the automorphism  $\varphi$  itself and by the inverse  $\varphi^{-1}$ . Therefore the automorphism  $\varphi$  induces an automorphism on each adjoint quotient  $F_l G / F_{l-1} G$ . On the other hand, by construction of the filtration, the automorphism  $\varphi$  acts on  $F_l G / F_{l-1} G$  as multiplication by  $2^l$ . This finishes the proof.  $\square$

Lemma 3.1 implies that following useful result.

**Proposition 3.2.** *Assume that a ring  $R$  contains  $\frac{1}{2}$ . Then the group  $TK_{n+1}^M(R)$  is uniquely 2-divisible.*

*Proof.* Consider a ring automorphism of  $R[\varepsilon]$  that sends  $\varepsilon$  to  $2\varepsilon$  and is identity on  $R$ . It induces an automorphism  $\varphi: K_{n+1}^M(R[\varepsilon]) \rightarrow K_{n+1}^M(R[\varepsilon])$ . Because of the equality  $1 + 2r\varepsilon = (1 + r\varepsilon)^2$ ,  $r \in R$ , Example 2.6(ii) implies that  $\varphi$  acts as multiplication by positive powers of 2 on symbols in  $TK_{n+1}^M(R)$ . By Lemma 3.1, this proves the proposition.  $\square$

Now we prove two lemmas on the Milnor  $K$ -group  $K_2^M(R[\varepsilon])$ .

**Lemma 3.3.** *Let  $R$  be any ring.*

(i) *For all elements  $a \in R^*$  and  $b \in R$  such that  $1 - a \in R^*$ , there is an equality in  $K_2^M(R[\varepsilon])$*

$$2 \left\{ 1 + \frac{b}{a} \varepsilon, 1 + \frac{b}{1-a} \varepsilon \right\} = 0.$$

(ii) *For all elements  $r_1, r_2 \in R^*$  such that  $r_1 + r_2 \in R^*$ , there is an equality in  $K_2^M(R[\varepsilon])$*

$$2\{1 + r_1 \varepsilon, 1 + r_2 \varepsilon\} = 0.$$

(iii) *Suppose that  $\frac{1}{2} \in R$  and  $R$  is weakly 4-fold stable. Then for all elements  $r_1, r_2 \in R$ , there is an equality in  $K_2^M(R[\varepsilon])$*

$$\{1 + r_1 \varepsilon, 1 + r_2 \varepsilon\} = 0.$$

*Proof.* (i) We have the Steinberg relation in  $K_2^M(R[\varepsilon])$

$$\{a + b\varepsilon, 1 - a - b\varepsilon\} = 0. \quad (2)$$

Note that

$$a + b\varepsilon = a \left( 1 + \frac{b}{a} \varepsilon \right), \quad 1 - a - b\varepsilon = (1 - a) \left( 1 - \frac{b}{1-a} \varepsilon \right). \quad (3)$$

Using multilinearity and the Steinberg relation  $\{a, 1 - a\} = 0$  in  $K_2^M(R)$ , we see that (2) and (3) imply the equality

$$\left\{ a, 1 - \frac{b}{1-a} \varepsilon \right\} + \left\{ 1 + \frac{b}{a} \varepsilon, 1 - a \right\} + \left\{ 1 + \frac{b}{a} \varepsilon, 1 - \frac{b}{1-a} \varepsilon \right\} = 0. \quad (4)$$

Applying the automorphism of  $R[\varepsilon]$  that sends  $\varepsilon$  to  $-\varepsilon$  and is identity on  $R$ , we get the equality

$$\left\{ a, 1 + \frac{b}{1-a} \varepsilon \right\} + \left\{ 1 - \frac{b}{a} \varepsilon, 1 - a \right\} + \left\{ 1 - \frac{b}{a} \varepsilon, 1 + \frac{b}{1-a} \varepsilon \right\} = 0. \quad (5)$$

Since  $(1 + r\varepsilon)^{-1} = 1 - r\varepsilon$  for any  $r \in R$ , the sum of (4) and (5) gives

$$2 \left\{ 1 + \frac{b}{a} \varepsilon, 1 - \frac{b}{1-a} \varepsilon \right\} = 0.$$

Taking the inverse element in the group  $K_2(R[\varepsilon])$ , we obtain item (i).

(ii) Apply item (i) with

$$a = \frac{r_2}{r_1 + r_2}, \quad b = \frac{r_1 r_2}{r_1 + r_2}.$$

(iii) Since  $(1 + s_2 \varepsilon) \cdot (1 + s_2 \varepsilon) = 1 + (s_1 + s_2) \varepsilon$  for all  $s_1, s_2 \in R$ , Example 2.3(i) implies that we may assume  $r_1, r_2 \in R^*$ . Since  $R$  is weakly 4-fold stable, there exists  $r \in R^*$  such that

$$r_1 + r, r_2 + r, \frac{r_1 + r_2}{2} + r \in R^*.$$

In particular, we have that  $r_1 + r_2 + 2r \in R^*$  and  $2r \in R^*$ . It follows from item (ii) that there are equalities

$$\begin{aligned} 0 &= 2\{1 + (r_1 + r)\varepsilon, 1 + (r_2 + r)\varepsilon\} = 2\{1 + r_1 \varepsilon, 1 + r_2 \varepsilon\} + 2\{1 + r \varepsilon, 1 + r_2 \varepsilon\} + \\ &\quad + 2\{1 + r_1 \varepsilon, 1 + r \varepsilon\} + 2\{1 + r \varepsilon, 1 + r \varepsilon\} = 2\{1 + r_1 \varepsilon, 1 + r_2 \varepsilon\}. \end{aligned}$$

Together with Proposition 3.2, this proves item (iii).  $\square$

**Lemma 3.4.** *Let  $R$  be a ring such that  $\frac{1}{2} \in R$  and  $R$  is weakly 4-fold stable. Let  $r_1, \dots, r_N \in R^*$ ,  $N \geq 2$ , be such that  $r_1 + \dots + r_N = 0$ . Then there is an equality in  $K_2^M(R[\varepsilon])$*

$$\{1 + r_1 \varepsilon, r_1\} + \dots + \{1 + r_N \varepsilon, r_N\} = 0.$$

*Proof.* We use induction on  $N$ . To prove the lemma for the case  $N = 2$ , observe that

$$\begin{aligned} \{1 + r_1 \varepsilon, r_1\} + \{1 + r_2 \varepsilon, r_2\} &= \{1 + r_1 \varepsilon, r_1\} + \{1 - r_1 \varepsilon, -r_1\} = \\ &= \{1 + r_1 \varepsilon, r_1\} - \{1 + r_1 \varepsilon, -r_1\} = \{1 + r_1 \varepsilon, -1\} \end{aligned}$$

and use Proposition 3.2.

To prove the lemma for the case  $N = 3$ , by the case  $N = 2$ , it is enough to show the equality

$$\{1 + (r_1 + r_2) \varepsilon, r_1 + r_2\} - \{1 + r_1 \varepsilon, r_1\} - \{1 + r_2 \varepsilon, r_2\} = 0.$$

This is equivalent to the equality

$$\{1 + (r_1 + r_2) \varepsilon, r_1 + r_2 - r_1 r_2 \varepsilon\} - \{1 + r_1 \varepsilon, r_1\} - \{1 + r_2 \varepsilon, r_2\} = 0, \quad (6)$$

because the symbol  $\{1 + (r_1 + r_2) \varepsilon, 1 - \frac{r_1 r_2}{r_1 + r_2} \varepsilon\}$  equals zero by Lemma 3.3(iii). Formula (6) is essentially proved by Suslin and Yarosh in [13, Lem. 3.5] (see also [10, Sublem. 3.3]). Namely, by multilinearity, the left hand side in (6) is equal to

$$\left\{ 1 + r_1 \varepsilon, 1 + \frac{r_2}{r_1} - r_2 \varepsilon \right\} + \left\{ 1 + r_2 \varepsilon, 1 + \frac{r_1}{r_2} - r_1 \varepsilon \right\}. \quad (7)$$

Applying the Steinberg relation twice, we obtain the equalities

$$\begin{aligned}
\left\{ 1 + r_1 \varepsilon, 1 + \frac{r_2}{r_1} - r_2 \varepsilon \right\} &= \left\{ (1 + r_1 \varepsilon) \left( -\frac{r_2}{r_1} + r_2 \varepsilon \right), 1 + \frac{r_2}{r_1} - r_2 \varepsilon \right\} = \\
&= \left\{ -\frac{r_2}{r_1}, 1 + \frac{r_2}{r_1} - r_2 \varepsilon \right\} = \left\{ -\frac{r_2}{r_1}, \left( 1 + \frac{r_2}{r_1} \right) \left( 1 - \frac{r_1 r_2}{r_1 + r_2} \varepsilon \right) \right\} = \\
&= \left\{ -\frac{r_2}{r_1}, 1 - \frac{r_1 r_2}{r_1 + r_2} \varepsilon \right\}.
\end{aligned}$$

Since the last expression is antisymmetric with respect to  $r_1$  and  $r_2$ , we obtain that (7) equals zero. This proves the case  $N = 3$ .

Now let us make the induction step for  $N \geq 4$ . For short, put  $\langle s \rangle := \{1 + s\varepsilon, s\}$ , where  $s \in R^*$ . The case  $N = 2$  asserts that  $\langle -s \rangle = -\langle s \rangle$  for any  $s \in R^*$ . The case  $N = 3$  asserts that  $\langle s_1 + s_2 \rangle = \langle s_1 \rangle + \langle s_2 \rangle$  for all  $s_1, s_2 \in R^*$  such that  $s_1 + s_2 \in R^*$ . Since  $R$  is weakly 4-fold stable, there is  $r \in R^*$  such that

$$r + r_1, r + r_1 + r_2, r + r_1 + r_2 + r_3 \in R^*.$$

We obtain the equalities

$$\begin{aligned}
\langle r_1 \rangle + \dots + \langle r_N \rangle &= -\langle r \rangle + \langle r \rangle + \langle r_1 \rangle + \dots + \langle r_N \rangle = \\
&= \langle -r \rangle + \langle r + r_1 \rangle + \langle r_2 \rangle + \dots + \langle r_N \rangle = \langle -r \rangle + \langle r + r_1 + r_2 \rangle + \langle r_3 \rangle + \dots + \langle r_N \rangle = \\
&= \langle -r \rangle + \langle r + r_1 + r_2 + r_3 \rangle + \langle r_4 \rangle + \dots + \langle r_N \rangle. \tag{8}
\end{aligned}$$

Expression (8) contains  $N - 1$  summands and all elements in angle parenthesis are from  $R^*$ . Besides, we have

$$(-r) + (r + r_1 + r_2 + r_3) + r_4 + \dots + r_N = 0.$$

Therefore, by the induction hypothesis, we obtain that (8) equals zero. This proves the lemma.  $\square$

Finally, recall the following result, which is proved in [11, Lem. 3.6] with a method of Nesterenko and Suslin from [12, Lem. 3.2] (see also Lemma 2.2 from the paper of Kerz [10]).

**Lemma 3.5.** *Let  $R$  be a weakly 5-fold stable ring. Then for all elements  $r, s \in R^*$ , there are equalities in  $K_2^M(R)$*

$$\{r, -r\} = 0, \quad \{r, s\} = -\{s, r\}.$$

Here, the second equality follows in a standard way from the first one and the identity

$$\{r, s\} + \{s, r\} = \{rs, -rs\} - \{r, -r\} - \{s, -s\}.$$



### 3.2 Proof of Theorem 2.9

If  $R$  is a weakly 2-fold stable ring, then by Example 2.3(i), the group  $\Omega_R^n$  is generated by differential forms  $sdr_1 \wedge \dots \wedge dr_n$ , where  $r_1, \dots, r_n \in R^*$ ,  $s \in R$ . The next lemma is similar to [1, Lem. 1.8] and the proof is straightforward.

**Lemma 3.6.** *Let  $R$  be a weakly 2-fold stable ring. Let  $M$  be an  $R$ -module and let  $f: (R^*)^{\times n} \rightarrow M$  be a map of sets such that*

(i)  *$f$  is alternating: for all  $r_1, \dots, r_n \in R^*$  and  $1 \leq i \leq n-1$ , we have*

$$f(r_1, \dots, r_i, r_{i+1}, \dots, r_n) = -f(r_1, \dots, r_{i+1}, r_i, \dots, r_n)$$

*and if  $r_i = r_{i+1}$ , then  $f(r_1, \dots, r_i, r_{i+1}, \dots, r_n) = 0$ ;*

(ii)  *$f$  satisfies the Leibniz rule: for all  $r_1, r'_1, r_2, \dots, r_n \in R^*$ , we have*

$$f(r_1 r'_1, r_2, \dots, r_n) = r_1 f(r'_1, r_2, \dots, r_n) + r'_1 f(r_1, r_2, \dots, r_n);$$

(iii)  *$f$  is additively multilinear: for all  $r_{11}, \dots, r_{1N}, r_2, \dots, r_n \in R^*$  (where  $N \geq 2$ ) such that  $r_{11} + \dots + r_{1N} = 0$ , we have*

$$f(r_{11}, r_2, \dots, r_n) + \dots + f(r_{1N}, r_2, \dots, r_n) = 0.$$

*Then there is a unique homomorphism of  $R$ -modules  $F: \Omega_R^n \rightarrow M$  such that for all  $r_1, \dots, r_n \in R^*$  and  $s \in R$ , we have that  $F(sdr_1 \wedge \dots \wedge dr_n) = sf(r_1, \dots, r_n)$ .*

Note that it is important to require in Lemma 3.6(iii) that  $N \geq 2$  is an arbitrary natural number, because the sum of invertible elements of a ring is not necessarily an invertible element. However for a weakly 4-fold stable ring  $R$ , the case of an arbitrary number  $N$  can be reduced to the cases  $N = 2, 3$ , which can be shown by the method from the end of the proof of Lemma 3.4.

In order to apply Lemma 3.6, we define an  $R$ -module structure on the group  $TK_{n+1}^M(R)$  (cf. [1, Lem. 1.4, 1.5]).

**Proposition 3.7.** *Let  $R$  be a weakly 5-fold stable ring such that  $\frac{1}{2} \in R$ . Then the following is true:*

(i) *The group  $TK_{n+1}^M(R)$  is generated by symbols of type  $\{1 + sr_1 \dots r_n \varepsilon, r_1, \dots, r_n\}$ , where  $r_1, \dots, r_n \in R^*$ ,  $s \in R$ .*

(ii) *The action of  $R$  on the ring  $R[\varepsilon]$  by endomorphisms that send  $\varepsilon$  to  $a\varepsilon$ , where  $a \in R$ , and are identity on  $R$  defines an  $R$ -module structure on  $TK_{n+1}^M(R)$  such that*

$$a : \{1 + sr_1 \dots r_n \varepsilon, r_1, \dots, r_n\} \longmapsto \{1 + asr_1 \dots r_n \varepsilon, r_1, \dots, r_n\}. \quad (9)$$

*Proof.* (i) It is easy to see that if  $R$  is weakly  $k$ -fold stable, then so is the ring  $R[\varepsilon]$ . Hence Lemma 3.5 holds for the weakly 5-fold stable ring  $R[\varepsilon]$ . Now combine this with Example 2.6(ii) and Lemma 3.3(iii).

(ii) The only non-trivial statement is distributivity with respect to elements from  $R$ , that is, the equality  $(a+b)v = av + bv$  for all  $a, b \in R$ ,  $v \in TK_{n+1}^M(R)$ . This follows from item (i) and formula (9).  $\square$

Formula (9) implies that the map  $B$  (see Definition 2.7) is a homomorphism of  $R$ -modules.

Now we are ready to prove the main result of the paper.

*Proof of Theorem 2.9.* Following the strategy from [1], we construct the inverse to the map  $B$ . By Proposition 3.7(ii), the group  $TK_{n+1}^M(R)$  is an  $R$ -module. Let us apply Lemma 3.6 to the map

$$f : (R^*)^{\times n} \longrightarrow TK_{n+1}^M(R), \quad (r_1, \dots, r_n) \longmapsto \{1 + r_1 \dots r_n \varepsilon, r_1, \dots, r_n\}.$$

For this, we need to check that  $f$  satisfies all conditions from Lemma 3.6. We use notation of this lemma. The map  $f$  is alternating by Proposition 3.2 and Lemma 3.5. The Leibniz rule for  $f$  follows from the equalities (see also formula (9))

$$\begin{aligned} & \{1 + r_1 r'_1 r_2 \dots r_n \varepsilon, r_1 r'_1, r_2, \dots, r_n\} = \\ &= \{1 + r_1 r'_1 r_2 \dots r_n \varepsilon, r'_1, r_2, \dots, r_n\} + \{1 + r_1 r'_1 r_2 \dots r_n \varepsilon, r_1, r_2, \dots, r_n\} = \\ &= r_1 \{1 + r'_1 r_2 \dots r_n \varepsilon, r'_1, r_2, \dots, r_n\} + r'_1 \{1 + r_1 r_2 \dots r_n \varepsilon, r_1, r_2, \dots, r_n\}. \end{aligned}$$

Finally, we show that  $f$  is an additively multilinear map. Indeed, by Lemma 3.4, there is an equality in  $K_2^M(R[\varepsilon])$

$$\{1 + r_{11} \varepsilon, r_{11}\} + \dots + \{1 + r_{1N} \varepsilon, r_{1N}\} = 0.$$

Applying the action of the element  $r_2 \dots r_n \in R$  (see formula (9)), we obtain the equality

$$\{1 + r_{11} r_2 \dots r_n \varepsilon, r_{11}\} + \dots + \{1 + r_{1N} r_2 \dots r_n \varepsilon, r_{1N}\} = 0.$$

Taking the product with the symbol  $\{r_2, \dots, r_n\} \in K_{n-1}^M(R)$ , we get the required equality.

Thus by Lemma 3.6, we obtain a homomorphism of  $R$ -modules  $F: \Omega_R^n \rightarrow TK_{n+1}^M(R)$  such that for all  $r_1, \dots, r_n \in R^*$  and  $s \in R$ , we have that

$$F(sdr_1 \wedge \dots \wedge dr_n) = \{1 + sr_1 \dots r_n \varepsilon, r_1, \dots, r_n\}. \quad (10)$$

By Proposition 3.7(i),  $TK_{n+1}^M(R)$  is generated by symbols from the right hand side of (10). Also,  $\Omega_R^n$  is generated by differential forms from the left hand side of (10). Therefore formula (10) together with Example 2.8 imply that the maps  $B$  and  $F$  are inverse to each other. This finishes the proof of Theorem 2.9.  $\square$

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